

U(∞) Gauge Theory from Higher Dimensions

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Abstract

We show that classical U(∞) gauge theories can be obtained from the dimensional reduction of a certain class of higher-derivative theories. In general, the exact symmetry is attained in the limit of degenerate metric; otherwise, the infinite-dimensional symmetry can be taken as spontaneously broken. Monopole solutions are examined in the model for scalar and gauge fields. An extension to gravity is also discussed.

1 Introduction

The idea of attributing the internal symmetry of fields to the symmetry of some compact space has grown in the last several decades. In recent years, many authors have studied the possibility of identifying the unified symmetry with the isometry of the extra spaces, which is motivated by Kaluza-Klein gravity [1] and superstring theory.[2] The local gauge symmetry in Kaluza-Klein theory originates from the symmetry of extra space.

The symmetry is only a part of an infinitely large symmetry of extra dimensions. In Ref. [3], Dolan and Duff advocated an interpretation that the large spatial symmetry is broken down to the symmetry of massless states by spontaneous compactification. It is natural that the infinite dimensional (Kac-Moody) symmetry is recovered in the infinite limit of compactification scale, or decompactification.

Here we show the simplest example. Suppose that the I -th dimension is compactified to a circle with circumference L . Then the general transformation of the off-diagonal component of metric is

$$\delta g_{\mu I} = \nabla_{\mu} \xi_I + \nabla_I \xi_{\mu} . \quad (1)$$

If we expand the parameter ξ_M ($M = \mu, I$) as

$$\xi_M = \sum_m \xi_M^m e^{imy/L} , \quad (2)$$

where y denotes the I -th coordinate, then the “zero mode” of (1) reduces to the gauge transformation of $A_\mu^0 \equiv g_{\mu I}^0$, i.e.

$$\delta A_\mu^0 = \nabla_\mu \xi_I^0, \quad (3)$$

while the transformation of the other field components are

$$\delta A_\mu^m = \nabla_\mu \xi_I^m + \frac{im}{L} \xi_\mu^m. \quad (4)$$

Therefore one can say that the “gauge” transformation is broken in the case of $m \neq 0$. The mass of the Kaluza-Klein excited state has the same origin as the right-most term of (4). The appearance of mass indicates symmetry breaking! The infinite-dimensional symmetry is restored in the decompactification limit, $L \rightarrow \infty$. For details, such as the algebraic structure of the field transformation, see Ref. [3].¹

The study of the possible structure of underlying symmetry is very useful in revealing the deeper mathematical importance of the model and to give a clear insight into model-building.

Recently Floratos et al. [5] offered a model with the gauge symmetry of an infinite dimensional group. The symmetry, $SU(\infty)$ is being eagerly investigated in the study of membrane theory.[6]

A naive expectation suggests that the $SU(\infty)$ model may be derived from some higher-dimensional model (at least in a certain limit), because the model contains an infinite number of particle states.

In this paper we take a heuristic approach. Note that the Lagrangian of the model is not necessarily of the same form as the dimensionally reduced one. For instance, the Lagrangian which leads to four-dimensional Yang-Mills theory does not have to be a Yang-Mills Lagrangian in higher dimensions.

The organization of this paper is as follows. In Sec. 2, we briefly review the $SU(\infty)$ Yang-Mills theory in order to make the present paper self-contained. In Sec. 3, we will illustrate a simple model which reduces to scalar field theory coupled to $SU(\infty)$ Yang-Mills fields. As we will see there, the $SU(\infty)$ Yang-Mills field comes from a $U(1)$ gauge field in higher dimensions. We will show that the two-dimensional $SU(\infty)$ Yang-Mills theory is obtained from the four-dimensional $U(1)$ model. In our models, the abelian gauge field remains as a zero mode after the dimensional reduction. Thus the symmetry group is actually $SU(\infty) \times U(1)$ or $U(\infty)$. In general dimensions, a D -dimensional $U(\infty)$ theory is derived from a $(D+2)$ -dimensional higher-derivative theory in the degenerate-metric limit. In Sec. 5, the Bogomol’nyi-Prasad-Sommerfield monopole [7] is constructed from the scalar and gauge theory introduced in Secs. 3 and 4. An $SU(2)$ subgroup is utilized in the construction. Section 6 is devoted to summary and outlook, including discussion of extension to a fermionic model and gravity.

¹For the symmetry breaking in higher dimensional Yang-Mills theory, see also Ref. [4].

2 Review of $SU(\infty)$ Yang-Mills Theory

It is well known that $SU(\infty)$ symmetry may be considered as a volume-preserving diffeomorphism of some two-dimensional surface, which can be identified with a membrane.[6]

The $SU(\infty)$ Yang-Mills theory [5] has been put forward in the context of the study of membrane theory.[6] To make our discussion self-contained, we review the $SU(\infty)$ Yang-Mills theory [5] in this section.

We consider $U(1)$ gauge fields which have dependence on “extra coordinates” θ and φ as well as the space-time coordinates. θ and φ span the standard coordinate system of the two-sphere. The gauge fields can be expanded in the following form:

$$A_\mu(x^\nu; \theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l A_\mu^{lm}(x^\nu) Y_{lm}(\theta, \varphi), \quad (5)$$

where $Y_{lm}(\theta, \varphi)$ is the spherical harmonic function.

$SU(\infty)$ field strength and gauge transformations are defined by

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}, \quad (6)$$

$$\delta A_\mu(x^\nu; \theta, \varphi) = \partial_\mu \omega(x^\nu; \theta, \varphi) + \{A_\mu, \omega\}, \quad (7)$$

$$\delta \tilde{F}_{\mu\nu} = \{\tilde{F}_{\mu\nu}, \omega\}, \quad (8)$$

where

$$\{f, g\} = \frac{\partial f}{\partial \cos \theta} \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial \cos \theta} = -\frac{1}{\sqrt{g^{(2)}}} \varepsilon^{mn} \partial_m f \partial_n g. \quad (9)$$

Here $g^{(2)}$ is the determinant of the standard metric of S^2 .

The bracket corresponds to the commutator in the usual Yang-Mills theory in the matrix-valued representation, while integration with respect to the extra coordinates corresponds to taking trace.

The algebra formed by the brackets can be identified with $SU(\infty)$ symmetry. This symmetry arises from the diffeomorphisms of S^2 . The diffeomorphisms of T^2 , the two-torus, can also lead to an $SU(\infty)$ algebra, although the process of taking the limit $N \rightarrow \infty$ in $SU(N)$ is slightly different from the case of S^2 . [8]

It is natural to suppose that such a theory is derived from some nonlinear theory in higher dimensions where θ and φ are the coordinates of extra spaces. In the next section, we will describe a model which leads to a scalar field coupled to the $SU(\infty)$ Yang-Mills field after the dimensional reduction as the simplest example.

3 Dimensional Reduction of a Scalar Model

In this section, we concentrate our attention on the derivation of classical scalar field theory with local $SU(\infty)$ symmetry from dimensional reduction.

The key step is to find the antisymmetric tensor in (9). We wish to consider that the tensor arises “spontaneously” in higher dimensions. To this end, we utilize the vacuum expectation value of the U(1) gauge field strength.

First, we consider the Lagrangian

$$L_S = \delta_{DEF}^{ABC} F^{DE} F_{AB} \partial^F \phi \partial_C \phi, \quad (10)$$

where the totally antisymmetric Kronecker’s symbol is defined by

$$\delta_{DEF}^{ABC} = 3! \delta_{[D}^A \delta_E^B \delta_{F]}^C, \quad (11)$$

and the generalization with more suffixes is straightforward. Our simple model for a scalar field is based on the Lagrangian (10).

The Lagrangian is rewritten as

$$L_S = 2(F^{AB} F_{AB} \partial^C \phi \partial_C \phi - 2F^{AC} F_{AB} \partial^B \phi \partial_C \phi). \quad (12)$$

Thus we can immediately see that the kinetic term of the scalar is generated if the field strength acquires an expectation value.

We consider the Lagrangian in $(D + 2)$ -dimensional space-time. We use indices m, n, \dots for compact two-dimensional space, say, S^2 or T^2 , while we use μ, ν, \dots for the D -dimensional space-time. For the present, we take a unit scale for the size of the compact space.

Here we assume a vacuum expectation value for the extra components of the field strength

$$\langle F_{mn} \rangle = \frac{1}{q} \varepsilon_{mn}, \quad (13)$$

where q is a constant. Note that $1/q$ is often quantized as an integer or half-integer (in the unit of the compactification volume) for topological reasons.[10]

Further we set $\partial_\mu A_m = 0$. If we leave A_m ’s, these will induce residual, nonminimal interacting scalars other than SU(∞) gauge fields, just as in Kaluza-Klein theory.

Classifying the suffices into μ ’s and m ’s, in the Lagrangian (10), we find

$$\begin{aligned} \frac{1}{2} \delta_{DEF}^{ABC} F^{DE} F_{AB} \partial^F \phi \partial_C \phi &= (F^{mn} F_{mn}) \partial^\mu \phi \partial_\mu \phi - 4F^{mn} F_{m\nu} \partial^\nu \phi \partial_n \phi \\ &+ 2(F^{m\mu} F_{m\mu} \partial^n \phi \partial_n \phi - F^{\mu m} F_{\mu n} \partial^n \phi \partial_m \phi) \\ &+ (F^{\mu\nu} F_{\mu\nu}) \partial^m \phi \partial_m \phi - 4F^{\mu m} F_{\mu\nu} \partial^\nu \phi \partial_m \phi \\ &+ 2(F^{m\mu} F_{m\mu} \partial^\nu \phi \partial_\nu \phi - F^{m\mu} F_{m\nu} \partial^\nu \phi \partial_\mu \phi) \\ &+ (F^{\mu\nu} F_{\mu\nu}) \partial^\lambda \phi \partial_\lambda \phi - 2F^{\mu\lambda} F_{\mu\nu} \partial^\nu \phi \partial_\lambda \phi. \end{aligned} \quad (14)$$

We first pay attention to the first line on the right-hand side of Eq. (14). Substitution of the ansätze, $F_{mn} = q^{-1} \varepsilon_{mn}$ and $\partial_\mu A_m = 0$ into this part yields

$$\begin{aligned} &(F^{mn} F_{mn}) \partial^\mu \phi \partial_\mu \phi - 4F^{mn} F_{m\nu} \partial^\nu \phi \partial_n \phi \\ &+ 2(F^{m\mu} F_{m\mu} \partial^n \phi \partial_n \phi - F^{\mu m} F_{\mu n} \partial^n \phi \partial_m \phi) \\ &\Rightarrow \frac{2}{q^2} (\partial_\mu \phi - q \varepsilon^{mn} \partial_m A_n \partial_n \phi)^2. \end{aligned} \quad (15)$$

This term is just the kinetic term of the scalar boson in the “adjoint” representation of $SU(\infty)$ symmetry,[9] $(D_\mu\phi)^2 = (\partial_\mu\phi + q\{A_\mu, \phi\})^2$. Note that a “neutral” scalar field is added, which comes from the zero mode of the expansion on the extra coordinates.

In general cases, the remaining terms in (14) do not have the $SU(\infty)$ local symmetry. Nevertheless, all the scalar modes are massless in the expansion on the extra coordinates, unlike ordinary Kaluza-Klein theory; this is because there is no $(\partial_m\phi\partial^m\phi)$ term in the Lagrangian.

Comparing the terms in the right-hand side of (14) with each other, we find a difference in the number of the extra-space indices contained in each term. The terms in the first line of the right-hand side contain two couples of repeating indices of m, n , while the second line contains one couple and the third none.

If we denote the radius of the extra space as b , instead of the unit scale, the metric has a dependence $g_{mn} \approx b^2$ and then $g^{mn} \approx b^{-2}$. Since the contraction of the extra-space indices is performed by use of g^{mn} , the first line, i.e. (15) becomes the dominant contribution for small b . At the same time, of course, we must rescale q so as to keep q/b^2 finite.

Therefore in the limit of the degenerate metric $b \rightarrow 0$ with an appropriate normalization of fields, we obtain an exact $SU(\infty)$ gauge symmetric model. Our model forms a good contrast to gauge theory from ordinary Kaluza-Klein gravity.[3] The latter exhibits large symmetry in the decompactification limit, while ours shows symmetric form in the degenerate compactification limit.

Our model has unbroken $SU(\infty)$ local symmetry in a particular dimension, even if the compactification scale is finite. This can easily be seen in three dimensional space-time; in this case the Lagrangian of our model can be written as

$$L_S \approx \left(\frac{1}{\sqrt{|g|}} \varepsilon^{ABC} F_{AB} \partial_C \phi \right)^2. \quad (16)$$

This gives only the contribution (15) when the dimensional reduction with the nonzero field strength is carried out.

In the next section, we will construct a model Lagrangian which induces the kinetic term for an $SU(\infty)$ Yang-Mills field.

4 $U(\infty)$ Yang-Mills Theory from Dimensional Reduction

As we have seen in the previous section, we can construct a model with local $SU(\infty)$ symmetry from dimensional reduction. In this section, we will write down the higher-dimensional Lagrangian which produces the $SU(\infty)$ Yang-Mills kinetic term by reduction.

We note that the models we are studying contain higher derivatives; moreover, the kinetic terms in D dimensions appear only if the field strengths take nonzero values. In this sense, our models can be compared to a gauge-theory version of a “pregeometric” theory of gravity.

Another point to notice is that the action of our model can be written in terms of differential forms.[11] For example, the action of the scalar model in the previous section is expressed as

$$\int (F \wedge d\phi) \wedge *(F \wedge d\phi), \quad (17)$$

where F is the $U(1)$ curvature two-form and $*$ denotes the dual.

Now, the model which leads to Yang-Mills must have a highly symmetric style. We can first generalize the Lagrangian (10) very naturally. Then we obtain

$$L_{YM} = \delta_{EFGH}^{ABCD} F^{EF} F_{AB} F^{GH} F_{CD}, \quad (18)$$

and this is proportional to

$$(F_{AB} F^{AB})^2 - 2 F^A{}_B F^B{}_C F^C{}_D F^D{}_A. \quad (19)$$

This form is also motivated by the Euler form in which $R_{ABCD} = F_{AB} F_{CD}$ is substituted. It is easy to rewrite the action in terms of differential forms as

$$\int (F \wedge F) \wedge *(F \wedge F), \quad (20)$$

which seems to be an extension of the Maxwell action

$$\int F \wedge *F. \quad (21)$$

In four dimensions, the Lagrangian (20) can be rewritten as

$$L_{YM} \approx \left(\frac{1}{\sqrt{|g|}} \varepsilon^{ABCD} F_{AB} F_{CD} \right)^2. \quad (22)$$

Note that this is not a “topological” Lagrangian even in four dimensions.

Here we take the same ansatz (13) for the field strength. This expectation value is consistent with the equation of motion in the vacuum. This is because the suffices are combined by Kronecker’s delta in the action and the excessive overlapping of the suffixes is avoided.

For the same reason, mass terms for the gauge fields are absent after dimensional reduction, unlike ordinary Kaluza-Klein type theories.

We first analyze the model in 2+2 dimensions, i.e. we adopt the Lagrangian (22). Assuming $\partial_\mu A_m = 0$ ($\mu = 0, 1$), we obtain

$$\begin{aligned} L_{YM}(D=2) &\approx \frac{1}{q^2} (\partial_0 A_1 - \partial_1 A_0 + q\{A_0, A_1\})^2 \\ &\equiv \frac{1}{2} \frac{1}{q^2} \tilde{F}_{01}^2, \end{aligned} \quad (23)$$

for compactification with a unit scale. Here we get the Lagrangian of two-dimensional classical $U(\infty)$ Yang-Mills theory.

The reason why we have called the symmetry group $U(\infty)$ rather than $SU(\infty)$ is as follows. For the $SU(\infty)$ Yang-Mills theory, the expansion by harmonics begins with $l = 1$ [see (5)]. In other words, we omit the zero mode of the two-manifold. In the derivation from higher dimensions, however, a $U(1)$ is included by the zero mode. Therefore, the gauge symmetry obtained after the reduction is $SU(\infty) \times U(1)$ or $U(\infty)$.

Next, in general dimensions, we get the $U(\infty)$ gauge theory from dimensional reduction in the degenerate limit, the scale of the extra space $b \rightarrow 0$, as in the scalar model in the previous section.

The classification of the Lagrangian done as in the previous section is as follows:

$$\begin{aligned}
(F_{AB}F^{AB})^2 - 2F^A{}_B F^B{}_C F^C{}_D F^D{}_A &= (F_{mn}F^{mn})^2 - 2F^m{}_n F^n{}_p F^p{}_q F^q{}_m \\
&+ 4(F_{mn}F^{mn})(F_{p\mu}F^{p\mu}) - 8F^\mu{}_m F^m{}_n F^n{}_p F^p{}_mu \\
&+ 2(F_{mn}F^{mn})(F_{\mu\nu}F^{\mu\nu}) - 8F^\mu{}_\nu F^\nu{}_m F^m{}_n F^n{}_\mu \\
&+ 4[(F^{m\mu}F_{m\mu})^2 - F^\mu{}_m F^m{}_\nu F^\nu{}_n F^n{}_\mu] \\
&+ 4(F_{m\mu}F^{m\mu})(F_{\nu\lambda}F^{\nu\lambda}) - 8F^\mu{}_\nu F^\nu{}_\lambda F^\lambda{}_m F^m{}_\mu \\
&+ (F_{\mu\nu}F^{\mu\nu})^2 - 2F^\mu{}_\nu F^\nu{}_\lambda F^\lambda{}_\sigma F^\sigma{}_\mu. \tag{24}
\end{aligned}$$

The first and second lines on the right-hand side of Eq. (24) vanish regardless of the dimension D , when the ansätze for the extra gauge fields are assumed. The third line becomes the Yang-Mills term $(4/q^2)\tilde{F}_{\mu\nu}^2$ when the ansätze are taken into consideration. The fourth and fifth lines remain if the compactification scale $b \neq 0$ and the dimension $D + 2 \neq 4$.

Here we consider generalization to more higher-derivative terms. Let us generalize the form (22) to a term including more derivatives, i.e. consider

$$(\varepsilon^{ABCDEFG\cdots} F_{AB} F_{CD} F_{EF} \cdots)^2. \tag{25}$$

This generic structure appears in the expansion of the determinant of some set of matrices including F_{AB} . These terms remind us of the Born-Infeld action,[12]²

$$S_{BI} = \int d^{D+2}x \frac{C}{\alpha^2} [\sqrt{-\det(\bar{g}_{AB} + \alpha F_{AB})} - \sqrt{-\bar{g}}], \tag{26}$$

where α is a coupling which has the dimensions of $(\text{mass})^{-2}$, while the constant C has the dimensions of $(\text{mass})^{D-2}$. This action can be expanded into terms like (25), with respect to small α .

Further we take the background geometry of partially compactified space as usual:

$$\bar{g}_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & b^2 \tilde{g}_{mn} \end{pmatrix}, \tag{27}$$

where b is the radius of compact space and μ, ν run over $0, 1, \dots, D$ while m, n denote the extra-coordinate indices as usual. \tilde{g}_{mn} is the metric on the extra

²For further references, see Ref. [13].

space with unit scale. If we take the field strength $F_{mn} = q^{-1}\varepsilon_{mn}$ as the background and set $\partial_\mu A_m = 0$ by hand as previously, we find the reduced Lagrangian in the limit of $b \rightarrow 0$ is proportional to

$$\frac{C}{\alpha^2} \left[\sqrt{-\frac{\alpha^2}{e^2} \det(g_{\mu\nu} + \alpha \tilde{F}_{\mu\nu})} \right], \quad (28)$$

where

$$\tilde{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + e\{A_\mu, A_\nu\}, \quad (29)$$

and $e = q/b^2$.

Now up to overall normalization and cosmological constant to be adjusted, we get the $U(\infty)$ Born-Infeld Lagrangian with the coupling constant e .

It is notable that if we first take the limit $\alpha \rightarrow 0$, we obtain only a usual Kaluza-Klein reduction of $U(1)$ fields.

In this section, we have considered the pure Yang-Mills sector of $U(\infty)$ theory. We will consider classical solutions in a $U(\infty)$ Yang-Mills-Higgs system in the following section.

5 BPS Monopole in the $U(\infty)$ Theory

Topological objects provide various views of field theory, both classical and quantum, and sometimes of the nonperturbative nature of it.

In the present section, we will discuss classical solutions in a $U(\infty)$ Yang-Mills-Higgs system.

In Sec. 3, we introduced a $U(\infty)$ scalar theory. Using this scalar as a Higgs field, we can consider a Yang-Mills-Higgs system. Although we can construct a potential term for the scalar field from dimensional reduction, we do so proceed in this paper.

The Lagrangian we consider is

$$\begin{aligned} L_{YMH} = & \frac{\beta}{4} (F^{AB} F_{AB} \partial^C \phi \partial_C \phi - 2F^{AC} F_{AB} \partial^B \phi \partial_C \phi) \\ & + \frac{\beta}{16} [(F_{AB} F^{AB})^2 - 2F^A{}_B F^B{}_C F^C{}_D F^D{}_A], \end{aligned} \quad (30)$$

where β is a coupling constant which has dimension $(\text{mass})^{-2}$.

We assume six-dimensional space-time. We must take care of the contribution of the nonminimal residual interaction which appears from the dimensional reduction.

We examine a static monopole configuration in the system. For simplicity, we investigate the Prasad-Sommerfield limit,[7] i.e. the limit of no Higgs-self-coupling. Thus we need not worry about the potential term.

To construct monopole configurations, we pick out an $SU(2)$ subgroup in the $U(\infty)$ group. This can easily be found; the $l = 1$ spherical harmonics $Y_{lm}(\theta, \varphi)$ ($m = -1, 0, 1$) play the role of generators of $SU(2)$, in the case of the algebra

formed by the bracket (9). Thus we assume that the gauge fields and scalar fields take nonzero values in this subgroup sector, i.e.

$$\phi = \sum_{a=1}^3 \phi^a(x^i) T^a(\theta, \varphi), \quad A_i = \sum_{a=1}^3 A_i^a(x^j) T^a(\theta, \varphi), \quad (31)$$

where T^a ($a = 1, 2, 3$) are defined as

$$T^1 = \frac{1}{\sqrt{2}}(Y_{11} + iY_{1-1}), \quad T^2 = \frac{1}{\sqrt{2}}(Y_{11} - iY_{1-1}), \quad T^3 = Y_{10}. \quad (32)$$

In (31), x^i [$i(j) = 1, 2, 3$] denotes the spatial coordinates. Besides these fields, of course, the “monopole” configuration of the gauge field [10] exists in the extra space, as Eq. (13).

Now the bracket is defined in terms of the extra coordinates with unit radius of the sphere. Then the general compactification scale b is absorbed into the coupling, $e = q/b^2 =$ a finite constant. If we require the condition of no physical singularity in the background gauge field which generates F_{mn} , e must be quantized as $2/n$ (n : integer).[10]

We make the following “spherical ansatz” for the solution in the explicit form:

$$\phi^a = \frac{x^a}{er^2} H(evr), \quad A_i^a = -\varepsilon_{aij} \frac{x^j}{er^2} [1 - K(evr)], \quad (33)$$

where $r = |x|$ and $a, i, j = 1, 2, 3$. v is an expectation value for $|\phi|$ which is taken at spatial infinity. In our case, ϕ gives a one-to-one mapping from a point of spatial infinity (S^2) to a point on the extra sphere S^2 .

H and K must be subject to the boundary condition

$$H(\xi) \rightarrow \xi, \quad K(\xi) \rightarrow 0 \quad \text{when } r \rightarrow \infty, \quad (34)$$

where $\xi = evr$.

For the time-independent ansatz, the energy of the system is obtained after integrating over the extra coordinates, θ and φ , as follows:

$$\begin{aligned} E = & \frac{8\pi^2 v \beta}{e^3 b^2} \int_0^\infty d\xi \left\{ \left(\frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left(\frac{dH}{d\xi} - \frac{H}{\xi} \right)^2 + \frac{(K^2 - 1)^2}{2\xi^2} + \frac{K^2 H^2}{\xi^2} \right. \\ & + \frac{(evb)^4}{30} \left[15H^2 \left(\frac{dK}{d\xi} \right)^2 + 20 \frac{1}{2} \left(\frac{dH}{d\xi} - \frac{H}{\xi} \right)^2 (1 - K)^2 + 18H^2(1 - K)^2 \right. \\ & \left. \left. + 14 \left(\frac{dK}{d\xi} \right)^2 (1 - K)^2 + 64\xi^2(1 - K)^4 \right] \right\}. \quad (35) \end{aligned}$$

At a glance, we can see the energy is always positive; moreover we find that the energy for finite b is larger than the mass of the Prasad-Sommerfield monopole, $E_0 = 8\pi^2 v \beta / (e^3 b^2)$ in the present case.

The field equation is obtained by the variational principle. If we consider the contribution of the second line of (35) as a perturbation, we expand the solution as $K = K_0 + \epsilon K_1 + \dots$ and $H = H_0 + \epsilon H_1 + \dots$, where $\epsilon = (evb)^4/30$. K_0 and H_0 are the solutions for the equation where we set $\epsilon = 0$, which satisfy the Bogomol'nyi equations.[7] We find the asymptotic behavior of K_1 and H_1 at spatial infinity and in the vicinity of the origin as follows:

$$\begin{aligned} K_1 &\approx e^{-\xi}, \quad \frac{H_1}{\xi} \approx \frac{1}{\xi} \quad \text{at spatial infinity,} \\ K_1 &\approx \xi^2, \quad H_1 \approx \xi^2 \quad \text{near the origin.} \end{aligned} \quad (36)$$

Unfortunately, the exact solution can be obtained only by numerical calculation. But here it is sufficient to know that the correction to the energy due to the residual interaction is of the order of ϵE_0 .

To summarize, we have found the monopole configuration in the system which is described by the higher-derivative action (30) in six dimensions. The solution is a nontrivial configuration of the gauge and scalar fields which belong to the SU(2) subgroup of the U(∞) gauge group which becomes an exact symmetry if $b = 0$. We find the energy of the monopole suffers the correction due to the residual Wteraeions if $b \neq 0$. The order of the correction is $\approx \epsilon E_0$, where $\epsilon = (evb)^4/30$.

Apparently, the lowest energy solution is attained if $b = 0$ for fixed β/b^2 . So we again see that $b = 0$ is a special point in our model.

As another example of a classical solution, we can consider locally concentrated magnetic flux using a Nielsen-Olesen vortex in $D + 2$ dimensions. We can construct the D -dimensional “string” [14] in which U(∞) Yang-Mills gauge bosons exist. The properties of this solution will be reported in a separate publication.

6 Summary and Outlook

To summarize : we have obtained the D -dimensional U(∞) Yang-Mills action from dimensional reduction of a $(D + 2)$ -dimensional theory. The partially “degenerate” metric tensor induces the large symmetry. This is compared with topological field theory,[15] in which the metric can be taken as completely degenerate. There is no massive excitation mode à la Kaluza-Klein regardless of the size of the extra space. The inverse of the coupling constant is quantized if we require regular background gauge fields in many cases.[10]

Our model Lagrangian is very similar to the term which appears in the expansion of the Born-Infeld Lagrangian.[12] If the Born-Infeld action is derived from (open) string theory,[16] α is related to the slope parameter. Therefore we can speculate that the large symmetry may have a relation to both the compactification scale and the “stringy” scale.

A monopole configuration has been considered. The lowest energy solution is realized when the residual interactions vanish, or $b = 0$.

Now we consider extensions to other fields. A possible extension to a fermion model which leads to coupling to the $SU(\infty)$ Yang-Mills field is represented as

$$L_F = i\delta_{DEF}^{ABC}\bar{\psi}\Gamma^{DEF}F_{AB}D_C\psi \approx i\bar{\psi}\Gamma^{ABC}F_{AB}D_C\psi, \quad (37)$$

where Γ^{ABC} is the antisymmetrized gamma matrix. This example is the simplest one which yields a minimal coupling to the $SU(\infty)$ Yang-Mills field after the reduction.

A naive extension to gravity is as follows:

$$\begin{aligned} L_G &= -\gamma\delta_{EFGH}^{ABCD}F^{EF}F_{AB}R^{GH}{}_{CD} \\ &\approx -\gamma(F^{CD}F_{AB}R^{AB}{}_{CD} - 4F^{CB}F_{AB}R_C^A + F^{AB}F_{AB}R), \end{aligned} \quad (38)$$

where γ is a coupling constant.

Regrettably, the Lagrangian has only one massless “graviton” even if the degenerate limit of the metric is taken. Thus the infinite-dimensional symmetry is always broken.

Nevertheless, it is known that the classical solution of the model described by the Lagrangian (38) has remarkable properties. In Ref. [17], Yoshida and the present author showed that the cosmological solutions derived from the Lagrangian are obtained in analytical forms for many cases of six-dimensional space-time. In the solutions, the inverse square of the radius of the extra two dimensions behaves similarly to the scale of the extra circle in five-dimensional Kaluza-Klein gravity. The correspondence holds in the static spherical solutions.[18] Whether this fact is connected to some deep physical insight or not is as yet obscure.

The Lagrangian (38) is very akin to the Euler form,

$$R^{CD}{}_{AB}R^{AB}{}_{CD} - 4R_A^C R_C^A + R^2. \quad (39)$$

Indeed the dimensional reduction of the Euler form yields (38) as a part of it.[19] It is known that six-dimensional Euler form gravity leads to an infinite number of massless gravitons in four dimensions if background space is partially compactified to $M_4 \times S^2$. [20, 21, 22] Studying the quantum aspects of the model in four dimensions is very attractive. We plan to analyze the Euler form gravity [20, 21, 22] and its quantum nature. A generic field theory in higher dimensions suffers from disastrous UV divergence. If the role of extra space is merely to generate a large symmetry, however, the quantum nature of the theory is expected to be remarkably improved. Both $1/D$ [23] and $1/N$ [24] expansion techniques may be available to study nonperturbative effects, where the number of gravitons is regularized to be N . Further we wonder if only one graviton remains massless and all the others become decoupled from physical spectra. We hope our model of $U(\infty)$ Yang-Mills theory may offer a toy model of such a higher-derivative theory.

The generalization to the case with higher-rank antisymmetric tensor can be considered. New types of symmetry may be reduced from such theories; it is an interesting possibility.

The models we have treated in this paper lead to classical $U(\infty)$ models after dimensional reduction. This statement is at the same level as in the case of five-dimensional Kaluza-Klein theory.[25] In the analyses of the previous sections we set $\partial_\mu A_m = 0$, while in Kaluza-Klein theory the pure Maxwell term appears if the radius of the extra space is set to be constant. Thus when we consider various aspects of our models we must be careful in treating the residual interactions of A_m . The treatment of zero modes of A_m will be reported elsewhere.

Note Added

After completion of this work, we became aware of the paper,[26] which is concerned with the $SU(\infty)$ gauge theory.

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